More Membrane Matrix Model Solutions, – and Minimal Surfaces in S^7

Joakim Arnlind and Jens Hoppe
Department of Mathematics
Royal Institute of Technology
Stockholm

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Abstract

New solutions to the classical equations of motion of a bosonic matrix-membrane are given. Their continuum limit defines 3-manifolds (in Minkowski space) whose mean curvature vanishes. Part of the construction are minimal surfaces in S^7, and their discrete analogues.

Some time ago [1], solutions of the bosonic matrix-model equations,

$$\begin{aligned}\ddot{X}_i &= -\sum_{j=1}^{d}\Big[\big[X_i, X_j\big], X_j\Big] \\ \sum_{i=1}^{d}&[X_i, \dot{X}_i] = 0\end{aligned} \tag{1}$$

were found where

$$X_i(t) = x(t)\mathcal{R}_{ij}(t)M_j, \tag{2}$$

with $\mathcal{R}(t) = e^{\mathcal{A}\varphi(t)}$ a real, orthogonal $d \times d$ matrix, $x(t)$ and $\varphi(t)$ being given via

$$\begin{gathered}\frac{1}{2}\dot{x}^2 + \frac{\lambda}{4}x^4 + \frac{L^2}{2x^2} = \text{const.} \\ \varphi^2(t)\dot{x}(t) = L(= \text{const}),\end{gathered} \tag{3}$$

and the d hermitean $N \times N$ matrices M_i satisfying

$$\begin{aligned}\sum_{j=1}^{d'}\Big[\big[M_i, M_j\big], M_j\Big] &= \lambda M_i \\ i &= 1, \ldots, d'.\end{aligned} \tag{4}$$

The reason for d' (rather than d) appearing in (4) was that in order to satisfy the two remaining conditions,

$$\mathcal{A}^2\overrightarrow{M} = -\overrightarrow{M} \tag{5}$$

$$\sum_{j=1}^{d}\big[M_j, (\mathcal{A}\overrightarrow{M})_j\big] = 0 \tag{6}$$

– which have to be fulfilled in order for (2) to satisfy (1) – in an "irreducible" way (the matrix valued d-component vector $\overrightarrow{M}$ can, of course, always be broken up to contain pairs of identical pieces) half – or more – of the matrices M_j were chosen to be zero, and (permuting the M's such that the first $d' \le \frac{d}{2}$ are the non-zero ones) the non-zero elements of $\mathcal{A}$ as $\mathcal{A}_{i+d',j} = 1 = -\mathcal{A}_{j,i+d'}$, $i, j = 1, \ldots, d'$; in particular, (6) was satisfied by having, for each j, either M_j or $(\mathcal{A}\overrightarrow{M})_j$ be identically zero.

As, in the membrane context. $d \overset{(<)}{=} 9$, $d' = 4$ recieved particular attention, while the continuum limit of (4),

$$\sum_j \{\{m_i, m_j\}, m_j\} = -\lambda m_i, \tag{7}$$

$$\left(\{m_i, m_j\} := \frac{1}{\rho}\big(\partial_1 m_i \partial_2 m_j - \partial_2 m_i \partial_1 m_j\big);\; g_{rs} := \partial_r \overrightarrow{m} \cdot \partial_s \overrightarrow{m};\; \overrightarrow{m} = \overrightarrow{m}(\varphi^1, \varphi^2)\right)$$

alias

$$\frac{1}{\rho}\partial_r \frac{g g^{rs}}{\rho}\partial_s \overrightarrow{m} = -\lambda \overrightarrow{m} \tag{8}$$

is related to

$$\frac{1}{\sqrt{g}}\partial_r\sqrt{g}g^{rs}\partial_s\vec{m} = -2\vec{m}, \qquad \vec{m}^2 = 1, \tag{9}$$

i.e the problem of finding minimal surfaces in higher dimensional spheres (which for $d' = 4$ was proven [2] to admit solutions of any genus).

In this letter, we would like to enlarge the realm of explicit solutions (of (1), resp. its $N \to \infty$ limit, resp (9)) while shifting emphasis from $d' = 4$ to $d' = 8$ (the case $d' = 6$, which can be used to obtain nontrivial solutions in the BMN matrix-model, will be discussed elsewhere).

Our first observation is that (6) rather naturally admits solutions which avoid the "doubling mechanism". While $\mathcal{A}$ is kept to be an "antisymmetric permutation"-matrix in a maximal even-dimensional space, (6) can be realized if $\mathbf{M} := \{M_j\}_{j=1}^d$ (with $M_d \equiv 0$ if d is odd) can be written as a union of even-dimensional subsets of mutually commuting members. In order to give a first example, let us, for later convenience, define (for arbitrary odd $N > 1$) N^2 independent $N \times N$ matrices

$$U_{\vec{m}}^{(N)} := \frac{N}{4\pi M(N)}\omega^{\frac{1}{2}m_1 m_2} g^{m_1} h^{m_2} \tag{10}$$

where $\omega := e^{\frac{4\pi i M(N)}{N}}$, $\vec{m} = (m_1, m_2)$,

$$g = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & \omega & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & \omega^{N-1} \end{pmatrix}, \quad h = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 & 1 \\ 1 & 0 & \cdots & 0 & 0 \end{pmatrix}. \tag{11}$$

(10) provides a basis of the Lie-algebra $\mathrm{gl}(N, \mathbb{C})$, with

$$\left[U_{\vec{m}}^{(N)}, U_{\vec{n}}^{(N)}\right] = -\frac{iN}{2\pi M(N)}\sin\left(\frac{2\pi M(N)}{N}(\vec{m}\times\vec{n})\right)U_{\vec{m}+\vec{n}}^{(N)} \tag{12}$$

(for the moment, we will put $M(N) = 1$, as only when $N \to \infty$, $\frac{M(N)}{N} \to \Lambda \in \mathbb{R}$, this "degree of freedom" is relevant).

Let now $N = 3$,

$$\begin{aligned}\vec{M} = \frac{1}{2}\Big(&\frac{U_{1,0}+U_{-1,0}}{2}, \frac{U_{1,0}-U_{-1,0}}{2i}, \frac{U_{0,1}+U_{0,-1}}{2}, \frac{U_{0,1}-U_{0,-1}}{2i}, \frac{U_{1,1}+U_{-1,-1}}{2}, \\ &\frac{U_{1,1}-U_{-1,-1}}{2i}, \frac{U_{-1,1}+U_{1,-1}}{2}, \frac{U_{-1,1}-U_{1,-1}}{2i}\Big) \\ =:& \big(M_1, M_2, M_3, M_4, M_5, M_6, M_7, M_8\big).\end{aligned} \tag{13}$$

(13) satisfies (4), $[M_1, M_2] = 0, [M_3, M_4] = 0, [M_5, M_6] = 0$ and $[M_7, M_8] = 0$ (note that we have implicitly reordered the elements of $\mathcal{A}$), and $\vec{M}^2 = \mathbb{1}$.

One can rewrite the 8 M_j's, being a basis of su(3), in terms of the Cartan-Weyl basis $\{h_1, h_2, e_\alpha, e_{-\alpha}, e_\beta, e_{-\beta}, e_{\alpha+\beta}, e_{-\alpha-\beta}\}$,

$$\begin{aligned}
&[h_1, h_2] = 0 \\
&[h_i, e_\alpha] = \alpha_i e_\alpha \quad [h_i, e_{-\alpha}] = -\alpha_i e_{-\alpha} \quad \alpha = (2, 0) \\
&[h_i, e_\beta] = \beta_i e_\beta \quad [h_i, e_{-\beta}] = -\beta_i e_{-\beta} \quad \beta = (-1, \sqrt{3}) \\
&[h_i, e_{\alpha+\beta}] = (\alpha+\beta)_i e_{\alpha+\beta} \quad [h_i, e_{-\alpha-\beta}] = -(\alpha+\beta)_i e_{-\alpha-\beta} \\
&[e_\alpha, e_{-\alpha}] = 4h_1 \qquad [e_\beta, e_{-\beta}] = -2h_1 + 2\sqrt{3}h_2 \\
&[e_{\alpha+\beta}, e_{-\alpha-\beta}] = 2h_1 + 2\sqrt{3}h_2 \\
&[e_\alpha, e_\beta] = 2e_{\alpha+\beta} \qquad [e_\alpha, e_{-\alpha-\beta}] = -2e_{-\beta} \\
&[e_{-\alpha}, e_{\alpha+\beta}] = 2e_\beta \qquad [e_{-\alpha}, e_{-\beta}] = -2e_{-\alpha-\beta} \\
&[e_\beta, e_{-\alpha-\beta}] = 2e_{-\alpha} \qquad [e_{-\beta}, e_{\alpha+\beta}] = -2e_\alpha,
\end{aligned} \tag{14}$$

obtaining

$$\begin{aligned}
M_1 &= \frac{3}{32\pi}\big(3h_1 + \sqrt{3}h_2\big) \\
M_2 &= \frac{3}{32\pi}\big(\sqrt{3}h_1 - 3h_2\big) \\
M_3 &= \frac{3}{32\pi}\big(e_\alpha + e_{-\alpha} + e_\beta + e_{-\beta} + e_{\alpha+\beta} + e_{-\alpha-\beta}\big) \\
M_4 &= \frac{3}{32\pi i}\big(e_\alpha - e_{-\alpha} + e_\beta - e_{-\beta} - e_{\alpha+\beta} + e_{-\alpha-\beta}\big) \\
M_5 &= \frac{3}{32\pi}\left(\sqrt{\omega}e_\alpha + \frac{1}{\sqrt{\omega}}e_{-\alpha} + e_\beta + e_{-\beta} + \sqrt{\omega}e_{\alpha+\beta} + \omega e_{-\alpha-\beta}\right) \\
M_6 &= \frac{3}{32\pi i}\left(\sqrt{\omega}e_\alpha - \frac{1}{\sqrt{\omega}}e_{-\alpha} + e_\beta - e_{-\beta} - \sqrt{\omega}e_{\alpha+\beta} + \omega e_{-\alpha-\beta}\right) \\
M_7 &= \frac{3}{32\pi}\left(\frac{1}{\sqrt{\omega}}e_\alpha + \sqrt{\omega}e_{-\alpha} + e_\beta + e_{-\beta} + \frac{1}{\sqrt{\omega}}e_{\alpha+\beta} + \frac{1}{\omega}e_{-\alpha-\beta}\right) \\
M_8 &= \frac{3}{32\pi i}\left(\frac{1}{\sqrt{\omega}}e_\alpha - \sqrt{\omega}e_{-\alpha} + e_\beta - e_{-\beta} - \frac{1}{\sqrt{\omega}}e_{\alpha+\beta} + \frac{1}{\omega}e_{-\alpha-\beta}\right)
\end{aligned} \tag{15}$$

where $\sqrt{\omega} = e^{\frac{2\pi i}{3}} = -\frac{1}{2} + \frac{\sqrt{3}}{2}i$ (in this equation, (15)).

By considering arbitrary *representations* of su(3) one can, also for higher N ($N \to \infty$), obtain a set of matrices,given by (15), satisfying (4), (5), (6).

When checking that (13) solves (4), one uses that, (N arbitrary)

$$\Big[\big[U^{(N)}_{\vec{m}}, U^{(N)}_{\vec{n}}\big], U^{(N)}_{-\vec{n}}\Big] = \frac{N^2}{4\pi^2}\sin^2\frac{2\pi}{N}\big(\vec{m}\times\vec{n}\big)U^{(N)}_{\vec{m}}, \tag{16}$$

and $\sin^2\frac{2\pi}{3} = \sin^2\frac{4\pi}{3}$.

Similarly, one may take

$$\begin{aligned}
\vec{M} = \frac{1}{2}\Bigg(&\frac{U_{\vec{m}} + U_{-\vec{m}}}{2}, \frac{U_{\vec{m}} - U_{-\vec{m}}}{2i}, \frac{U_{\vec{m}'} + U_{-\vec{m}'}}{2}, \frac{U_{\vec{m}'} - U_{-\vec{m}'}}{2i}, \\
&\frac{U_{\vec{n}} + U_{-\vec{n}}}{2}, \frac{U_{\vec{n}} - U_{-\vec{n}}}{2i}, \frac{U_{\vec{n}'} + U_{-\vec{n}'}}{2}, \frac{U_{\vec{n}'} - U_{-\vec{n}'}}{2i}\Bigg),
\end{aligned} \tag{17}$$

with

$$\vec{m}' = \begin{pmatrix} -m_2 \\ m_1 \end{pmatrix} \qquad \vec{n}' = \begin{pmatrix} -n_2 \\ n_1 \end{pmatrix},$$

which is a solution of (4) for $N = \hat{N} := \vec{m}^2 + \vec{n}^2$ (which we assume to be odd), write the M_j's (8 $\hat{N} \times \hat{N}$ matrices) as ($\hat{N}^2$-dependent) linear combinations of a ($\hat{N}$ "independent") basis of $\mathrm{gl}(\hat{N}, \mathbb{C})$

$$M_j^{(\hat{N})} = \sum_{a=1}^{\hat{N}^2-1} \mu_j^a(\hat{N}) T_a^{(\hat{N})}, \tag{18}$$

$$\left[T_a^{(\hat{N})}, T_b^{(\hat{N})}\right] = i f_{ab}^c T_c^{(\hat{N})} \tag{19}$$

and then define

$$M_j^{(N)} := \sum_{a=1}^{\hat{N}^2-1} \mu_j^a(\hat{N}) T_a^{(N)} \tag{20}$$

to obtain corresponding solutions for $N > \hat{N}$ (by letting $T_a^{(N)}$ be N-dimensional representations of (19)).

In the case of $\vec{m}^2$ being equal to $\vec{n}^2$, this detour is not necessary, and (17) *directly* gives solutions of (4) for *any* (odd) N. The reason is that, by using (16) the "discrete Laplace operator"

$$\Delta_{\vec{M}}^{(N)} := \sum_{j=1}^{d} \Big[\big[\,\cdot\, , M_j \big], M_j \Big], \tag{21}$$

when acting on any of the components of $\vec{M}$, in each case yields the same scalar factor ("eigenvalue")

$$\frac{N^2}{4\pi^2} \left(\sin^2 \frac{2\pi}{N} (\vec{m} \times \vec{n}) + \sin^2 \frac{2\pi}{N} \vec{m}^2 + \sin^2 \frac{2\pi}{N} (\vec{m} \cdot \vec{n}) \right). \tag{22}$$

The $N \to \infty$ limit of this construction gives (a solution of (7))/(8), resp. (9))

$$\begin{aligned} \vec{m}(\varphi^1, \varphi^2) = \frac{1}{2} \Big(& \cos \vec{m}\vec{\varphi}, \sin \vec{m}\vec{\varphi}, \cos \vec{m}'\vec{\varphi}, \sin \vec{m}'\vec{\varphi}, \\ & \cos \vec{n}\vec{\varphi}, \sin \vec{n}\vec{\varphi}, \cos \vec{n}'\vec{\varphi}, \sin \vec{n}'\vec{\varphi} \Big), \end{aligned} \tag{23}$$

which for each choice

$$\vec{m} = \begin{pmatrix} m_1 \\ m_2 \end{pmatrix} \quad \vec{n} = \begin{pmatrix} n_1 \\ n_2 \end{pmatrix} \quad \vec{m}' = \begin{pmatrix} -m_2 \\ m_1 \end{pmatrix} \quad \vec{n}' = \begin{pmatrix} -n_2 \\ n_1 \end{pmatrix} \quad \vec{m}^2 = \vec{n}^2$$

describes a minimal torus in S^7.

Interestingly, the $N \to \infty$ limit, (23), allows for non-trivial deformations (apart from the arbitrary constant that can be added to each of the 4 different arguments), namely

$$\begin{aligned} \vec{m}_\gamma = \frac{1}{2} \Big(& \cos\gamma \cos \vec{m}\vec{\varphi}, \cos\gamma \sin \vec{m}\vec{\varphi}, \cos\gamma \cos \vec{m}'\vec{\varphi}, \cos\gamma \sin \vec{m}'\vec{\varphi}, \\ & \sin\gamma \cos \vec{n}\vec{\varphi}, \sin\gamma \sin \vec{n}\vec{\varphi}, \sin\gamma \cos \vec{n}'\vec{\varphi}, \sin\gamma \sin \vec{n}'\vec{\varphi} \Big). \end{aligned} \tag{24}$$

It is easy to check that (24) solves (9) (and (8), with an appropriate choice of ρ, constant), but when "checking" (7) (which is *identical* to (8)) via the $N \to \infty$ limit of (12), the γ-dependence of the m_j at first looks as if leading to a "contradicition" (it *would*, in the finite N-case), but the rationality of the structure-constants ($\overrightarrow{m} \times \overrightarrow{n}$ instead of $\frac{N}{2\pi} \sin \frac{2\pi}{N} (\overrightarrow{m} \times \overrightarrow{n})$) comes at rescue.

To come to the final observation of this note, rewrite (24) as

$$\overrightarrow{m}_\gamma = \frac{1}{\sqrt{2}} \vec{x}_+^{[\gamma]} + \frac{1}{\sqrt{2}} \vec{x}_-^{[\gamma]} \tag{25}$$

with

$$\begin{aligned} \vec{x}_+^{[\gamma]} = \frac{1}{2} \Big(& \cos(\overrightarrow{m}\overrightarrow{\varphi} + \gamma), \sin(\overrightarrow{m}\overrightarrow{\varphi} + \gamma), \cos(\overrightarrow{m}'\overrightarrow{\varphi} + \gamma), \sin(\overrightarrow{m}'\overrightarrow{\varphi} + \gamma), \\ & \sin(\overrightarrow{n}\overrightarrow{\varphi} + \gamma), -\cos(\overrightarrow{n}\overrightarrow{\varphi} + \gamma), \sin(\overrightarrow{n}'\overrightarrow{\varphi} + \gamma), -\cos(\overrightarrow{n}'\overrightarrow{\varphi} + \gamma) \Big) \\ \vec{x}_-^{[\gamma]} = \frac{1}{2} \Big(& \cos(\overrightarrow{m}\overrightarrow{\varphi} - \gamma), \sin(\overrightarrow{m}\overrightarrow{\varphi} - \gamma), \cos(\overrightarrow{m}'\overrightarrow{\varphi} - \gamma), \sin(\overrightarrow{m}'\overrightarrow{\varphi} - \gamma), \\ & -\sin(\overrightarrow{n}\overrightarrow{\varphi} - \gamma), \cos(\overrightarrow{n}\overrightarrow{\varphi} - \gamma), -\sin(\overrightarrow{n}'\overrightarrow{\varphi} - \gamma), \cos(\overrightarrow{n}'\overrightarrow{\varphi} - \gamma) \Big) \end{aligned} \tag{26}$$

While γ, in this form, becomes irrelevant (insofar each of the 4 arguments in $\vec{x}_+ := \vec{x}_+^{[0]}$, as well as those in $\vec{x}_- := \vec{x}_-^{[0]}$ can have an arbitrary phase-constant), not only their sum, (25), but (due to the mutual orthogonality of $\vec{x}_+, \partial_1\vec{x}_+, \partial_2\vec{x}_+, \vec{x}_-, \partial_1\vec{x}_-$ and $\partial_2\vec{x}_-$) *both* $\vec{x}_+$ and $\vec{x}_-$ *separately*, in fact any linear combination

$$\vec{x}_\theta = \cos\theta \vec{x}_+ + \sin\theta \vec{x}_- \tag{27}$$

gives a minimal torus in S^7.

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References

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